Islamic Mathematics

Overview

Islamic Mathematics is the term used to refer to the mathematics done in the Islamic world between the 8th and 13th centuries CE. Mathematics from the medieval Middle East is very important to the mathematics we use today. While Europe endured its “Dark Ages,” the Middle East preserved and expanded the arithmetic, geometry, trigonometry, and astronomy from the ancient Greek philosophers, such as Euclid. The most important contribution may be the invention of algebra, which originated in Baghdad in the House of Wisdom (*bayt al-hikma*).

House of Wisdom

Norman MacDonald/Saudi Aramco World/PADIA

The House of Wisdom was primarily a library and a place for translation and research. Scholars would work here in translating Greek and Hindu treatises to Arabic, and also conducted their own research and wrote original treatises. The House of Wisdom was established in the early 9th century, by Caliph al-Rashid. His son, Caliph al-Ma’mun, was the ruler who made the House of Wisdom so important. Al-Ma’mun had a dream in which Aristotle appeared to him; after this dream, al-Ma’mun wanted to translate as many Greek manuscripts as he could! He commissioned scholars to begin translating Greek, Hindu, Syriac-Persian, and Hebrew texts into Arabic. Most of these texts dealt with philosophy or mathematics and science.

Al-Khwārizmī

Muhammad ibn Mūsā al-Khwārizmī is probably the most famous Muslim mathematician. He lived about 800-847 CE. Al-Khwārizmī was born in Qutrubull, an area near Baghdad between the Tigris and Euphrates rives, but was brought to work at the House of Wisdom by the Caliph al-Ma’mun. He popularized a number of mathematical concepts, including the use of Hindu-Arabic numbers and the number zero, algebra, and the use of geometry to demonstrate and prove algebraic results. Many of his works deal with astronomy, but he also wrote about the Jewish calendar, arithmetic, and algebra.

Picture of al- Khwārizmī on a Russian stamp, issued in 1983

Arithmetic

Hindu-Arabic Numbers	Arabic-language Numbers
0	٠
1	١
2	٢
3	٣
4	٤
5	٥
6	٦
7	٧
8	٨
9	٩

Al-Khwārizmī wrote a very important treatise on Hindu-Arabic numerals, which made the use of these numbers popular. The introduction of the number zero was especially important for mathematics, and the number 0 was used for about 250 years throughout the Islamic world before Europe ever heard of it! He also introduced the Hindu concept of decimal positioning notation to the Arab and European worlds, which we still use today!

Algebraic Operations

Al-Khwārizmī wrote a treatise entitled *Kitab al-jabr wa'l-muqabalah*. The treatise actually had a very practical reason behind it: the longest chapter of the treatise teaches people how to apply algebra to Islamic inheritance laws! The words *al-jabr* and *al-muqabalah* were operations used by Al-Khwārizmī, much like addition, subtraction, multiplication, and division. *Al-jabr* means something like "restoration" or "completion," and was the operation used to add equal terms to both sides of an equation to get rid of a negative term.

For example, with the equation

$$x^2 = 40x - 4x^2,$$

al-Khwārizmī uses *al-jabr* to add $4x^2$ to both sides of the equation, getting the result:

$$5x^2 = 40x.$$

He can then complete the problem by division

$$x^2 = 8x$$
$$x = 8$$

Though we now know $x = 0$ & 8, Al-Khwārizmī never allows a variable to equal zero.

Al-muqabalah means something like "balancing," and was the operation used to subtract equal terms from both sides of an equation. For example, al-Khwārizmī has the equation:

$$50 + x^2 = 29 + 10x,$$

so he uses *al-muqabalah* to subtract 29 from each side, getting the result:

$$21 + x^2 = 10x.$$

From here, al-Khwārizmī can then complete the problem:

$$x^2 - 10x + 21 = 0$$

$$(x - 7)(x - 3) = 0$$

$$x = 3, 7$$

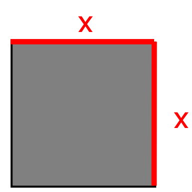
As you can see, *al-muqabalah* and *al-jabr* were operations defined by al-Khwārizmī which we still use today, though we don't call them the same thing! His operation *al-jabr*, adding equal amounts to both sides of the equation, is where our word "algebra" comes from!

Completing the Square: I

Al-Khwārizmī also created the method that we now call "completing the square." Al-Khwārizmī literally completed the square! If he had an equation:

$$x^2 + 10x = 39,$$

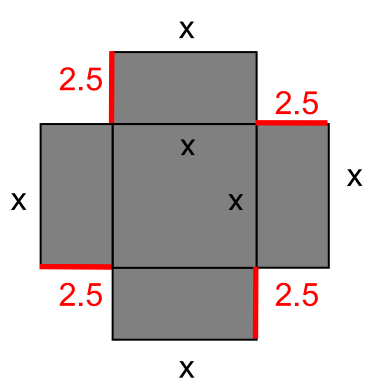
al-Khwārizmī would create the following figure. First he would make a small square, with each side length equal to x.



This square then has an area of

$$x^2 = 39 - 10x.$$

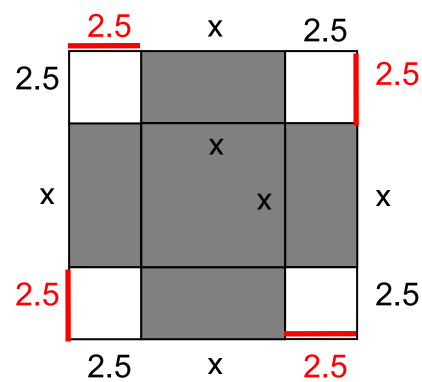
Then, he added on rectangles to each side of the square, each with a width of $^{10}/_4$ (= 2.5).



So, al-Khwārizmī knows that the area of the inner square plus the four rectangles (the figure above) is equal to 39. In other words,

$$x^2 + 4(2.5x) \quad = \quad x^2 + 10x \quad = \quad 39$$

Completing the figure to make a large square, with corner squares with sides equal to 2.5, he then had a figure he could use to find x!



Therefore, the area of the large square, with sides equal to $x + 5$, is equal to the area of the shaded region plus the area of each corner square:

$$39 + 4(2.5)^2 \quad = \quad 39 + 25 \quad = \quad 64.$$

So, $$(x + 5)^2 = 64,$$

and al-Khwārizmī can then find his result:

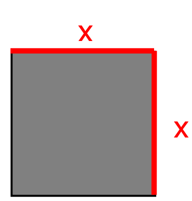
$$(x + 5) = 8, -8$$
$$x = 3, -13.$$

Completing the Square: II

Al-Khwārizmī also provides a second way to "complete the square," which might be easier for modern algebra students to use. Using the same equation:

$$x^2 + 10x = 39,$$

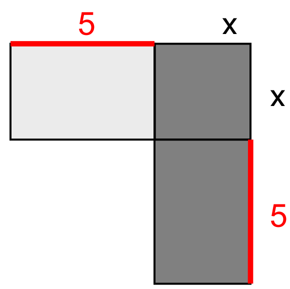
he creates the following figure. First, we make the square with sides equal to x.



Again, the area of this square is

$$x^2 = 39 - 10x.$$

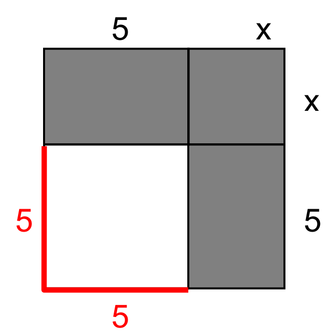
Then, he added on rectangles with lengths $^{10}/_2$ (= 5).



So, the area of this figure (the square and two rectangles) is equal to 39. In other words:

$$x^2 + 2(5x) \quad = \quad x^2 + 10x \quad = \quad 39$$

He then completed the figure to make a large square by finishing the square with sides of length equal to 5, which gave a figure that can be used to find x!



Then, al-Khwārizmī uses the same method he used above to get the area of the completed large square, with sides equal to x + 5:

$$(x^2 + 10x) + 5^2 \quad = \quad 39 + 25 \quad = \quad (x+5)^2 = 64$$

And he gets the same result as he did above:

$$(x+5) = 8, -8$$

$$x = 3, -13.$$

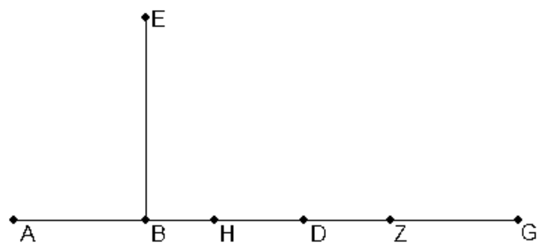
So, when we talk about "completing the square" when we solve quadratic equations, we refer to the method of al-Khwārizmī which literally created and completed a square to find the value of x!

Geometry: Ibn Sinān

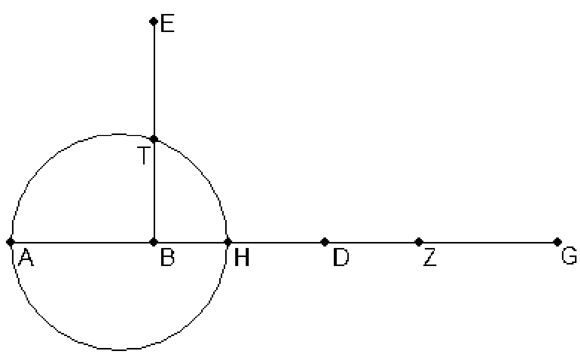
Ibrāhīm ibn Sinān (d. 946) is the grandson of Thābit ibn Qurra, the famous mathematician and translator of Archimedes. His treatment of the area of a segment of a parabola is the simplest construction from the time before the Renaissance. He wrote that he invented the proof out of necessity, to save his family's scientific reputation after hearing accusations that his grandfather's method was too long-winded! He also was more concerned with general methods and theories than with particular problems.

Parabola

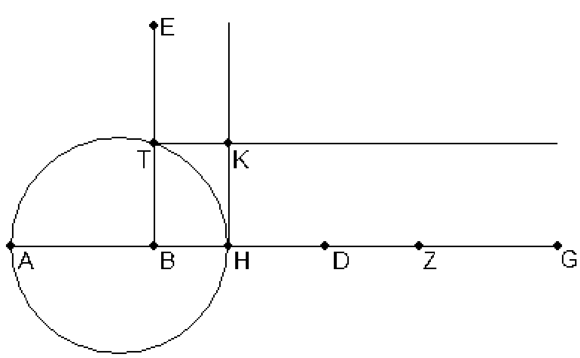
Ibn Sinān describes the following method for drawing a parabola. First, draw a line AG. Create a fixed segment AB on AG and construct BE perpendicular to AB. On BG, pick as many points H, D, Z …as you want.



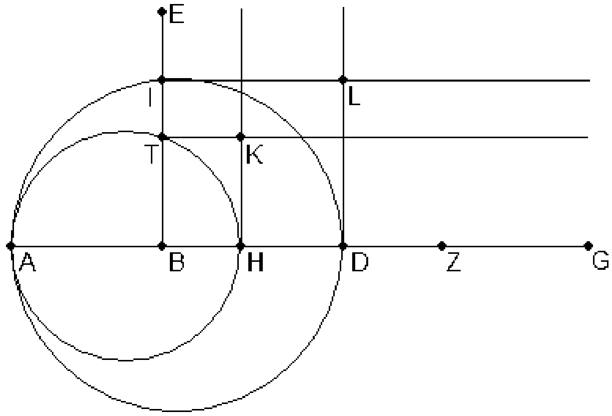
Starting with the point H, create a circle with diameter AH, and let the perpendicular BE intersect it at T.



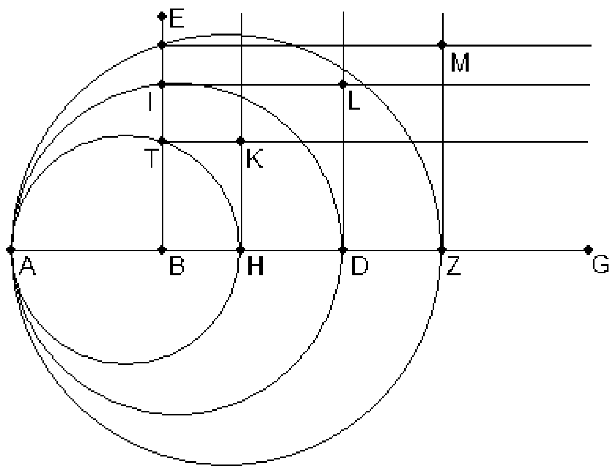
Through T, create a line parallel to AB and through H, draw a line parallel to BE. These lines intersect at K.



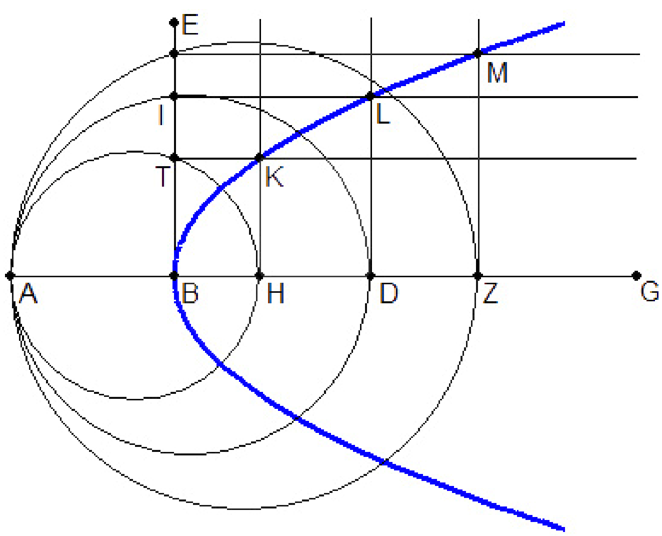
Now, draw a circle with diameter AD, and let this intersect BE at I. Following the same procedure as above, draw a line through I parallel to AG and a line through D parallel to BE. Let these lines intersect at L.



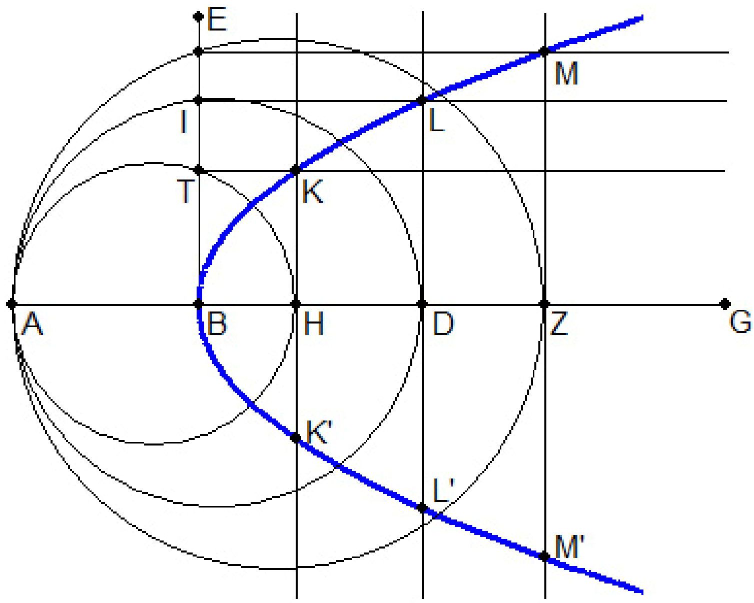
Follow the same construction method for the remaining points Z … to obtain the corresponding intersection points M …



Then these points B, K, L, M, … lie on the parabola with vertex B, axis BG, and parameter AB!



Create K', L', M' …on the extensions of the lines KH, LD, MZ … so that KH = HK', LD = DL', MZ = ZM' … Then, these points K', L', M' … also lie on the parabola!



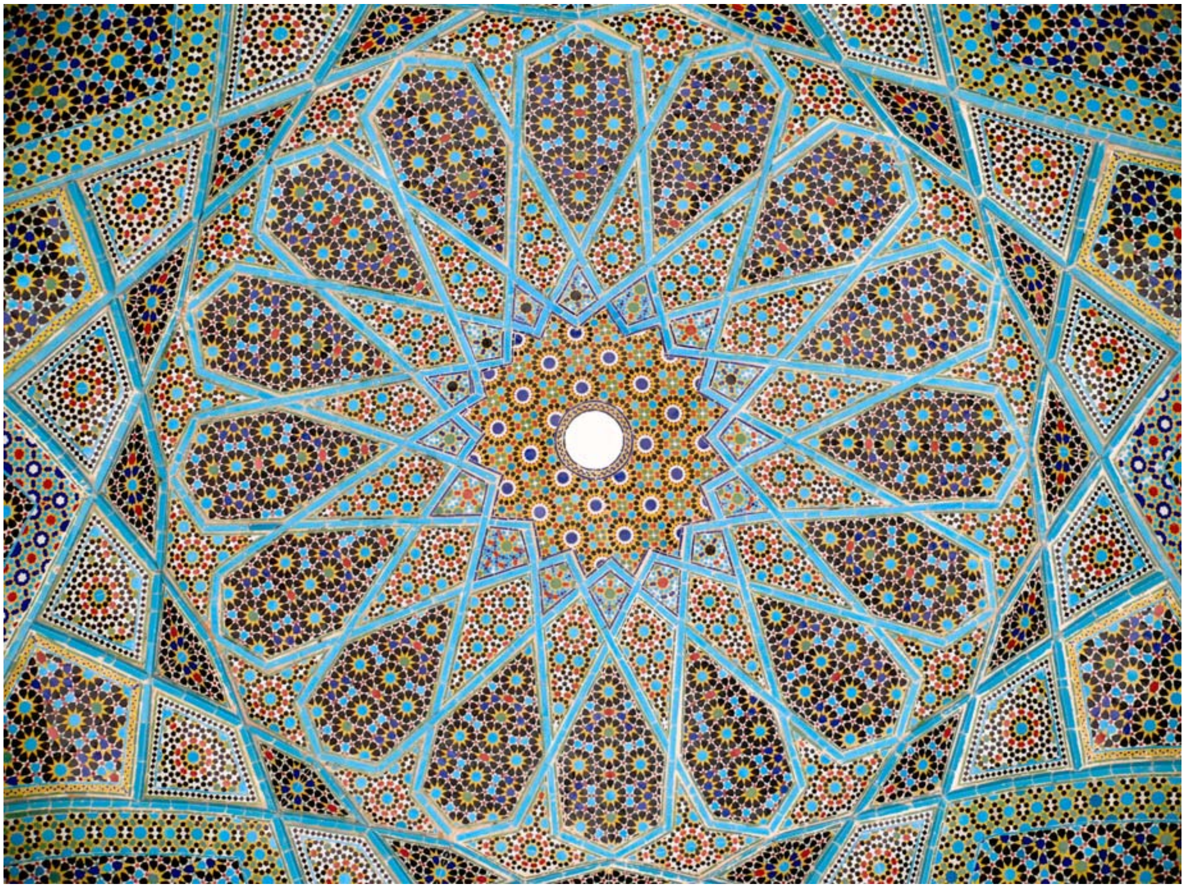
Ibn Sinān also proves that K is on the parabola with a proof by contradiction:

- He assumes that the parabola does not pass through K, which means that it must pass through another point N on KH. So, $NH^2 = AB \cdot BH$, by the property of the parabola.
- However, since TB is perpendicular to the diameter of the semicircle ATH, he points out that $TB^2 = AB \cdot BH$ (by a rule from Euclid's *Elements*).
- Further, he has constructed TBHK to be a parallelogram, so TB = KH.
- So, $KH^2 = TB^2 = AB \cdot GB = NH^2$ which means that KH = NH and K = N, which contradicts his first assumption.
- Therefore, K lies on the parabola.

Ibn Sinān applies the same proof to L, M … to prove the validity of his parabolic construction.

This method shows the ability of Ibn Sinān and Muslim mathematicians to construct a proof in the style of the Greeks, as well as their contributions to geometry by providing more concise geometrical constructions and proofs.

Geometry in Art



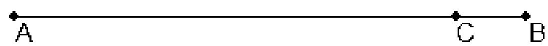
Pentocelo/Roof of the tomb of Persian poet Hafez at Shiraz, Iran, Province of Fars/Wikimedia

Much of the geometry done in the Islamic world was concerned with flat, 2-dimensional, figures, and was primarily concerned with the practical uses of geometry in art. Because Islamic art created flat designs and patterns, in opposition to the perspective art which emerged in the West in the "renaissance" period, Islamic geometry did not develop to the modern geometry which describes surfaces and 3-dimentional figures.

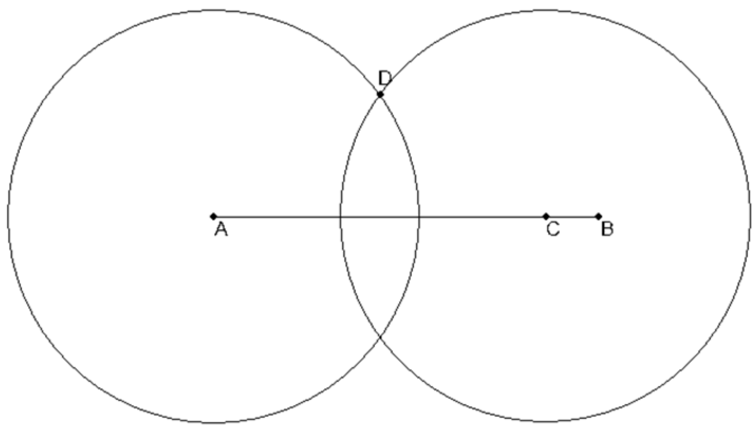
Islamic art used a lot of geometry, so mathematicians such as **Abu Nasr al-Farabi** wrote treatises on how to solve geometrical common problems for artists. Abu Nasr al-Farabi taught philosophy in both Baghdad and Apello (in northern Syria), and was killed by highway robbers outside Damascus in 950 CE. He wrote a treatise called *A Book of Spiritual Crafts and Natural Secrets in the Details of Geometrical Figures*, in which he taught artists how to solve the following three problems, which helped artist produce infinite patterns in their designs and to fit their patterns into specific areas.

Problem 1: To construct at the endpoint A of a segment AB a perpendicular to that segment, without extending the segment beyond A.

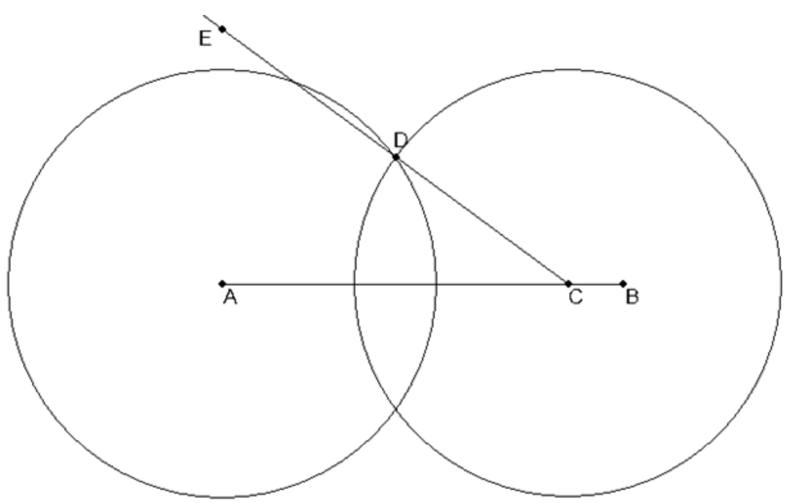
Procedure: On AB mark a segment AC.



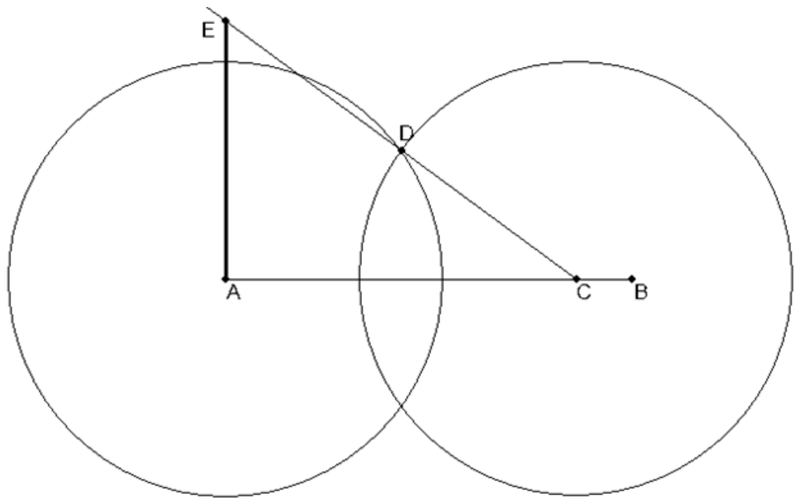
Now, draw circles centered at A and C, which meet at D.



Extend CD beyond D to E so that ED = DC.



Then angle CAE is a right angle.

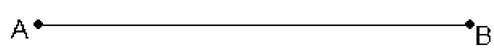


Proof:

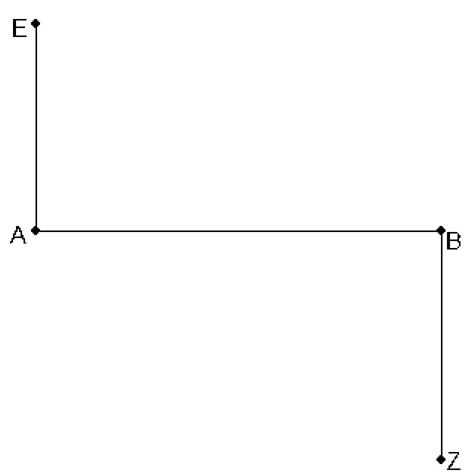
- The circle that passes through E; A; C has D as a center, since DC = DA = DE.
- So, EC is a diameter of that circle.
- Therefore, angle EAC is an angle in a semicircle.
- Hence, angle EAC is a right angle.

Problem 2: To divide a line segment into any number of equal parts; for example, three equal segments. In other words, we will divide the line segment AB into the equal parts AG = GD = DB.

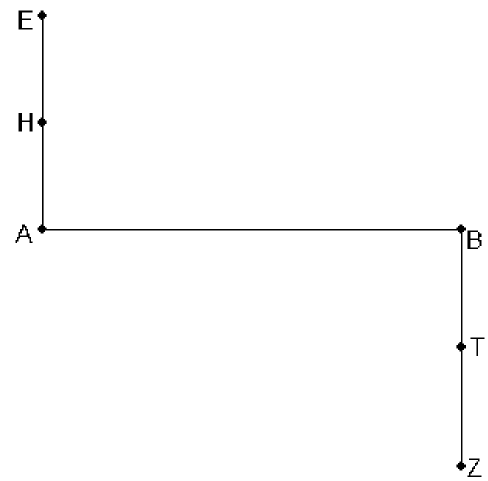
Procedure: Begin with line segment AB:



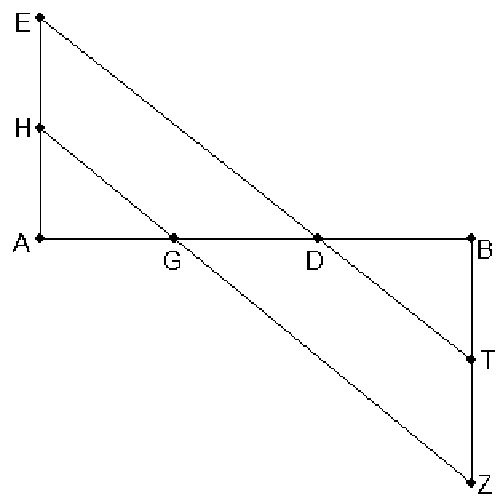
At both endpoints create perpendiculars AE = BZ in opposite directions.



On the perpendiculars, add midpoints H and T so that AH = HE = BT = TZ.



Now join H to Z and E to T by straight lines which cut AB at G; D respectively. Then AG = GD = DB.

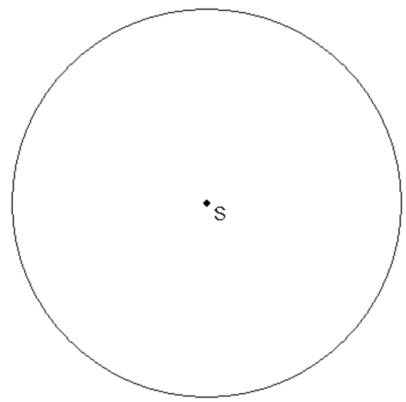


Proof:

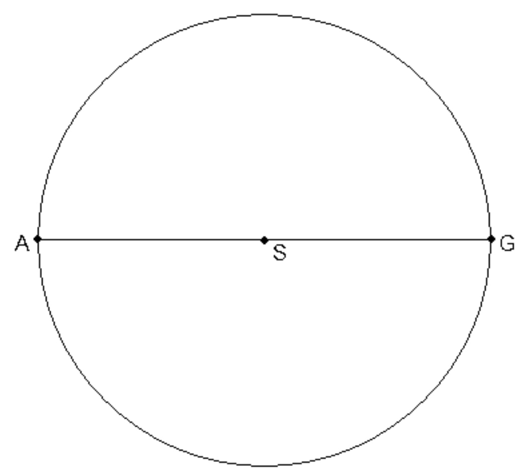
- AHG and BTD are two right triangles with equal angles at G and D (and therefore at H and T).
- Also, HA = BT.
- So, the triangles are congruent and AG = BD.
- Because HG and ED are parallel, triangles AHG and AED are similar.
- So, AH / AG = (AH + HE) / (AG + GD).
- This means: AH (AG + GD) = AG (AH + HE).
- Which yields: AH (GD) = AG (HE).
- But, EH = HA and so:
- DG = GA.

Problem 3: To construct a square in a given circle.

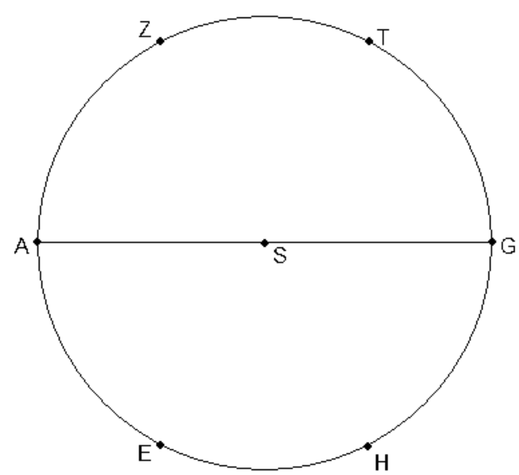
Procedure: Given circle S:



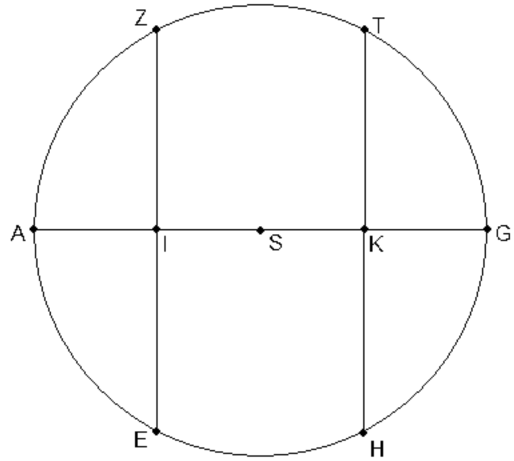
Locate the center S and draw a diameter ASG.



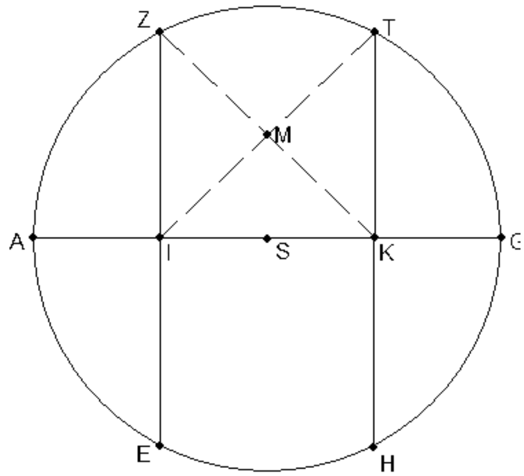
Create equal arcs AZ, AE, GT, and GH



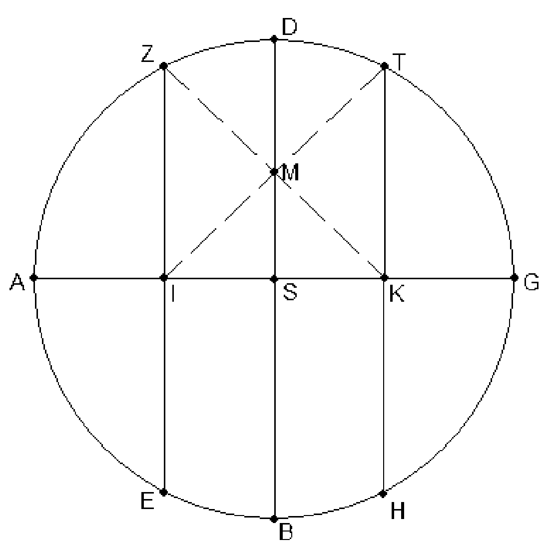
Draw lines ZE and TH, which intersect the diameter at I and K.



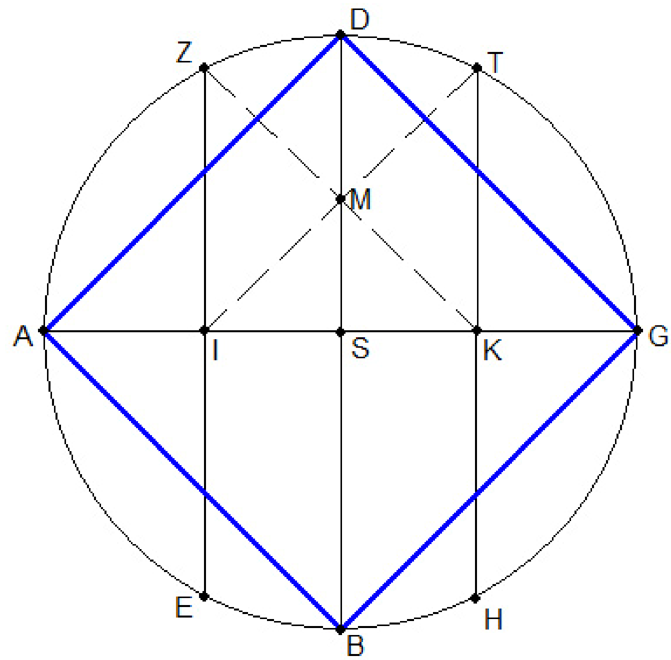
Draw ZK and TI, which intersect at M.



Draw the diameter through S, M. Let it meet with the circle at D and B.



Then ADGB will be a square.



Proof:

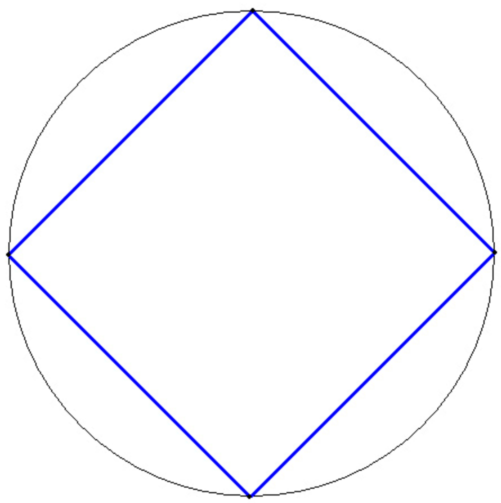
- Since arcZA = arcAE, the diameter GA bisects arcZE.
- So, GA is perpendicular to line segment ZE, the chord of arcZE.
- Similarly, GA is perpendicular to TH;
- So angle TKI and angle ZIK are both right angles.
- Since TH and ZE are chords of equal arcs, they are equal.
- Therefore their halves, TK and ZI, are equal.
- Because since TK and ZI are also parallel (both perpendicular to GA) the figure TKIZ is a rectangle.
- Its diagonals ZK and TI therefore are equal and bisect each other.
- So MK = MI; in other words, triangle MKI is isosceles.
- Since the equal chords ZE and TH are equidistant from the center (KS = SI).
- So in the isosceles triangle MKI, the line MS bisects the side KI and is therefore perpendicular to KI.
- Thus the diameter DB is perpendicular to the diameter GA.
- Therefore, ADGB is a square.

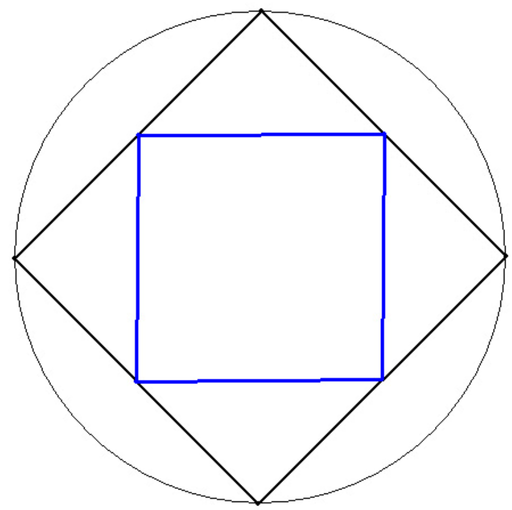
The Islamic art of geometrical design led to an expansion of geometry proofs and constructions; geometry was improved to help the creation of beautiful geometric Islamic cultural art. These three example problems show that the Muslim geometers like al-Farabi were very concerned with the practical applications of their work. In this, they were similar to al-Khwārizmī, whose algebra was created to help Muslims adhere to Islamic inheritance laws. Though modern mathematics is mainly concerned with mathematical theory, Islamic mathematics was developed for practical applications (both physical, such as art, and social, such as inheritance laws).

Art Patterns

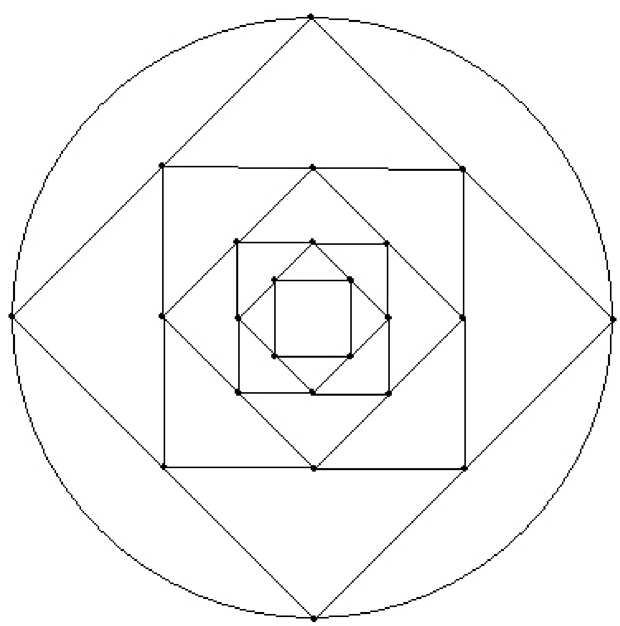
Artists had to learn how to solve these problems in order to create intricate patterns. These patterns often used all of the methods described above. For example:

Starting with a circumscribed square (obtained by the methods above), divide each side into two equal parts (by the method described above). Then connect these midpoints to form another square inside the first one.

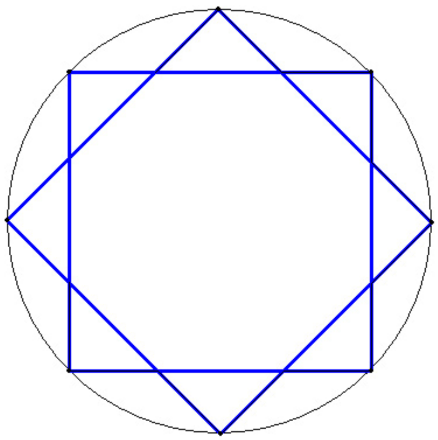


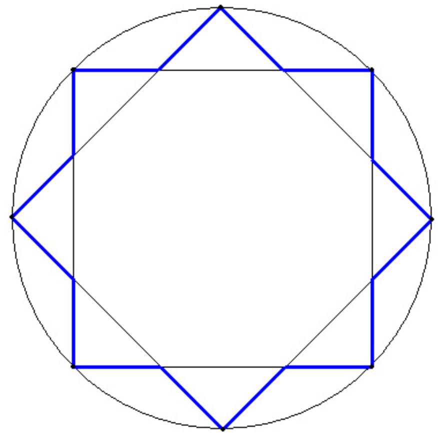


Repeat this, creating new squares inside each of your previous square. Then, your completed pattern would look like this:

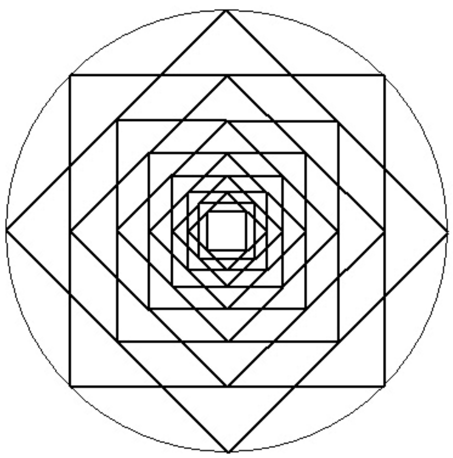


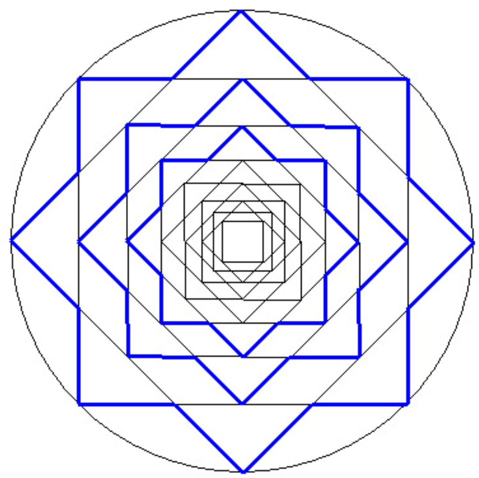
Artists used similar circumscribed squares to make other designs as well. An 8-pointed star was made by creating two circumscribed squares.

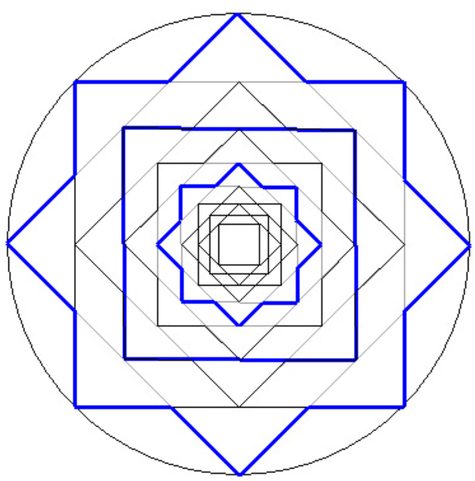




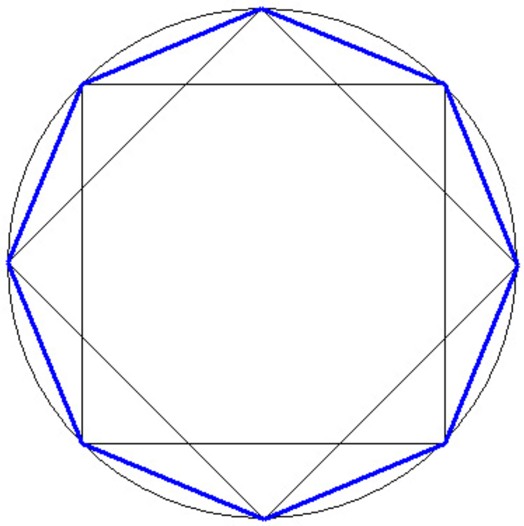
Continuing the same pattern as above (divide each side into two equal lengths, and connect these midpoints to create smaller squares), artists could get the more complex patterns:

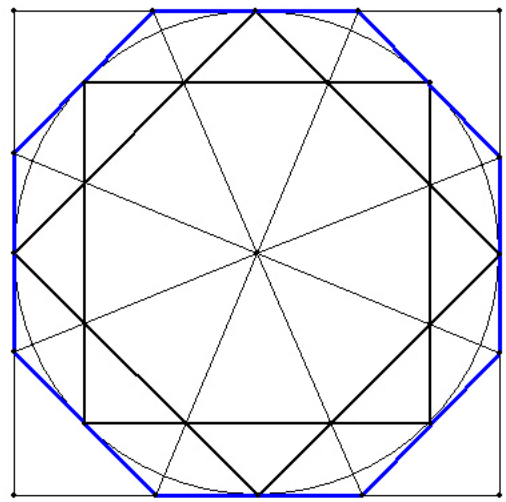




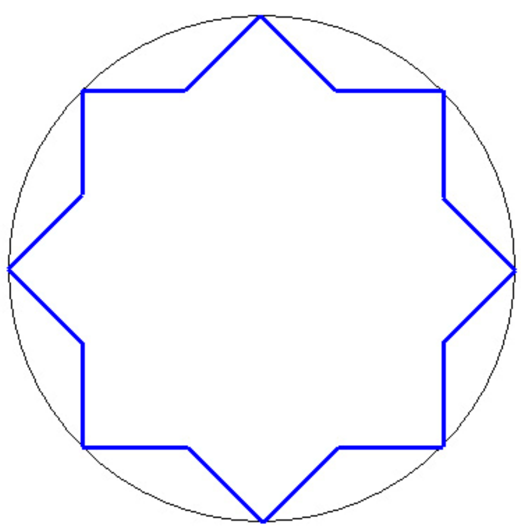


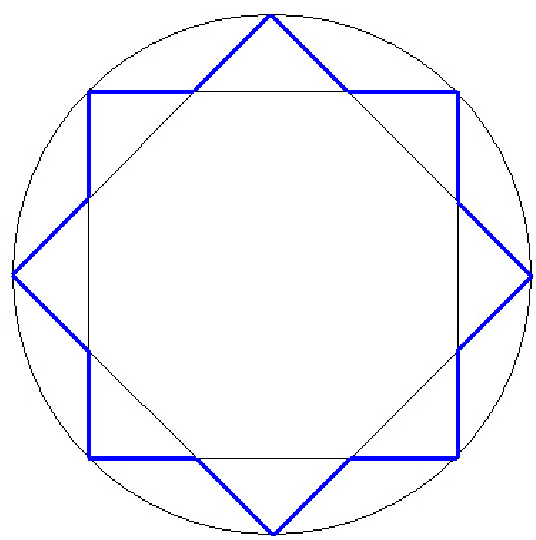
Connecting the endpoints of each circumscribed square, artists could create circumscribed octagons (below, left). Artists also made squares around the circles, and used these sides and tangent lines to create octagons (below, right).



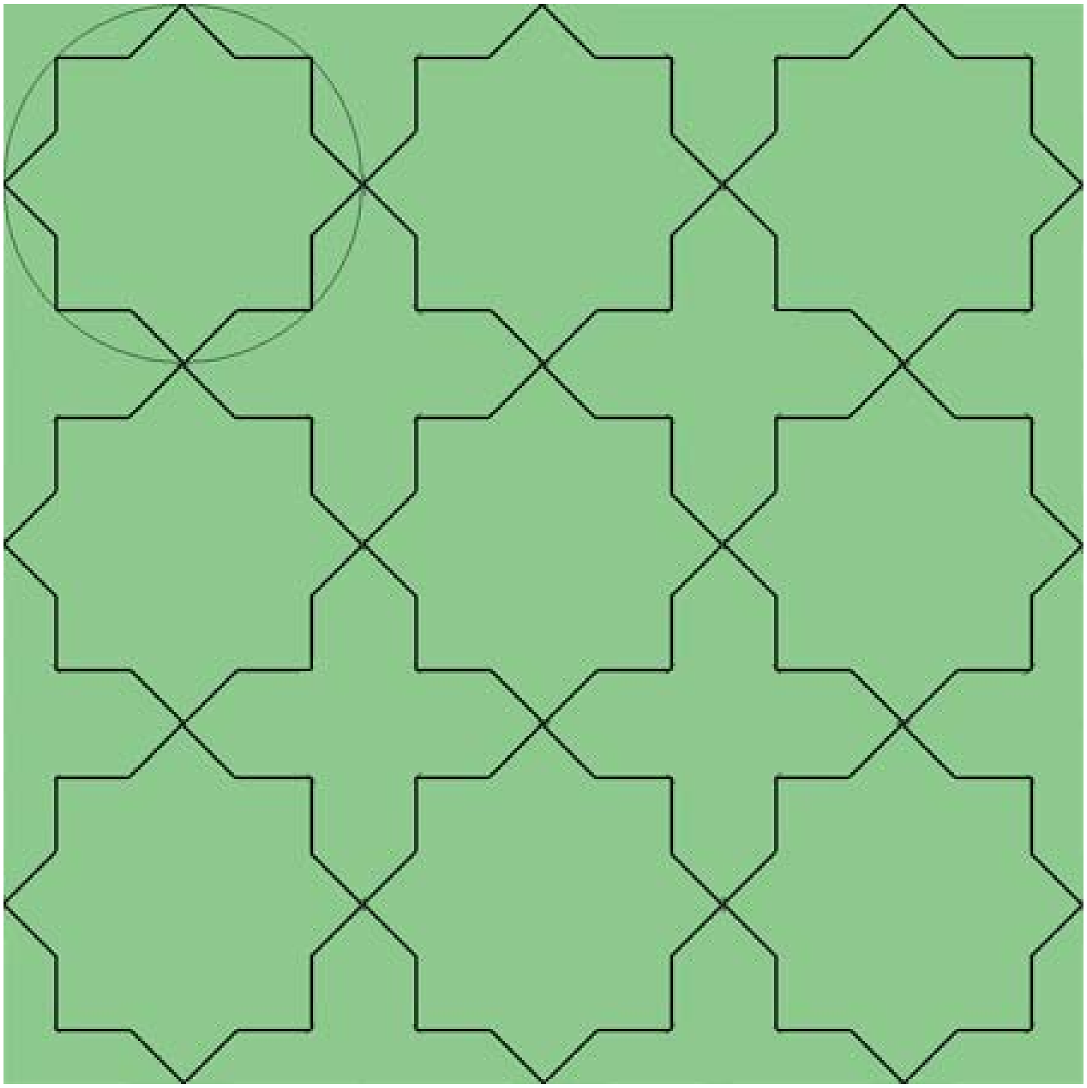


Infinite patterns were created by connecting these shapes together. For example, using the eight-pointed star that artists made by creating two circumscribed squares by the methods above:

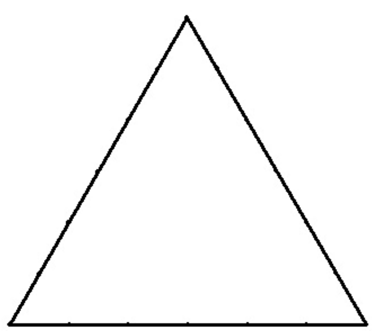




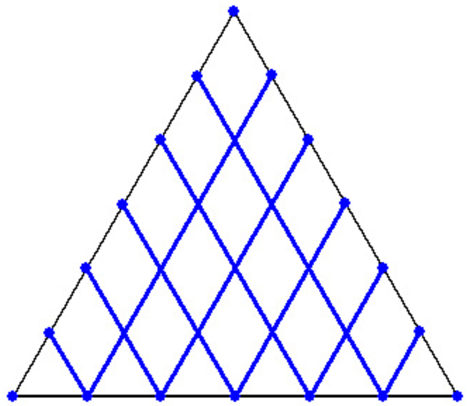
Artists could then create the following pattern:



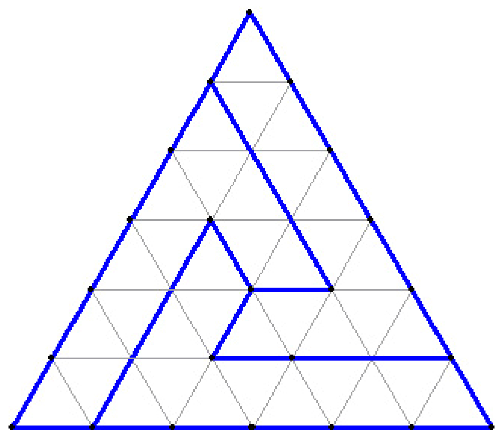
Artists also created borders with infinite patterns. For example, consider the following design. First, begin with an equilateral triangle:



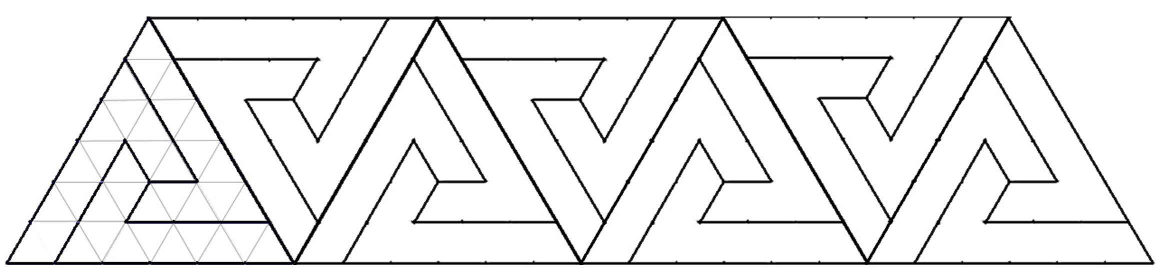
Now, divide each side into six equal parts, using the method described above:



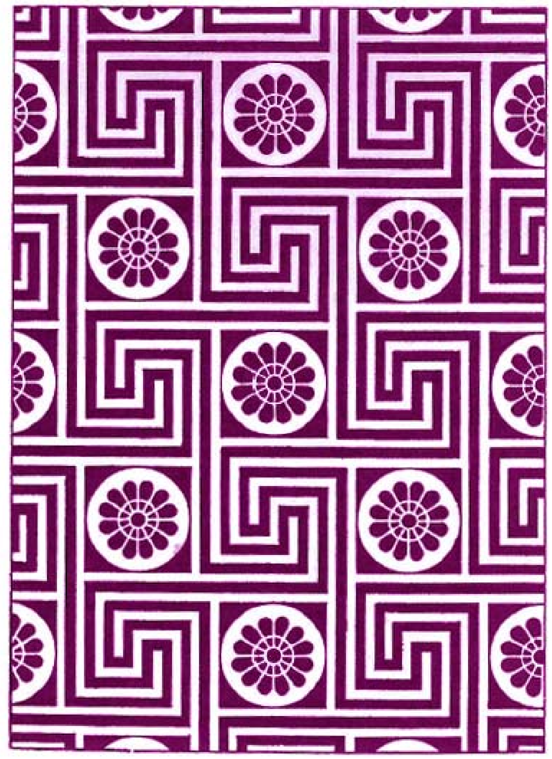
Then, trace the following pattern along the diagonal lines which were used to divide each side into seven equal parts:

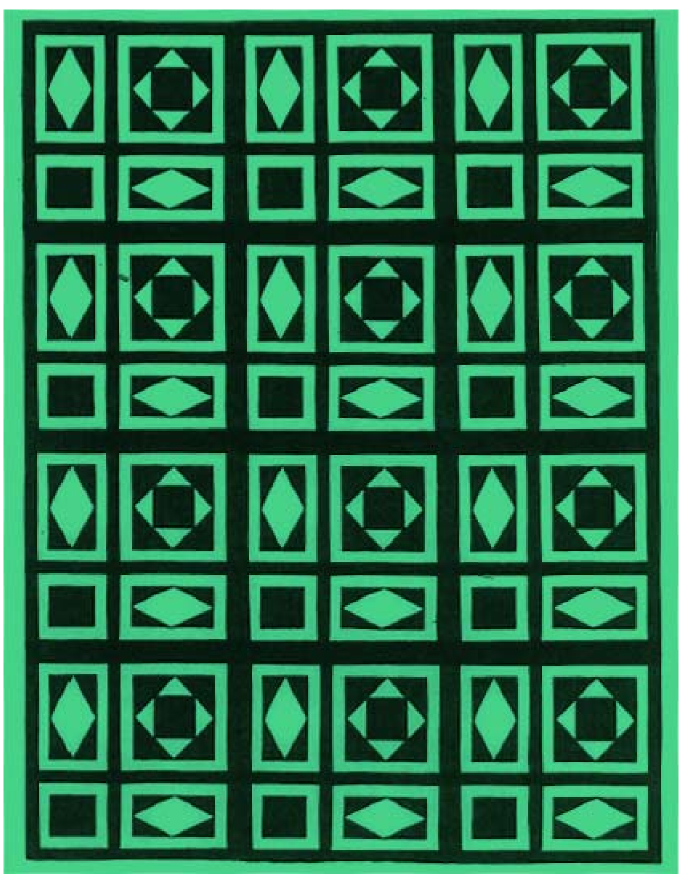


Now, this triangle can be laid side-by-side, rotating every-other triangle by 180 degrees, to create a border:



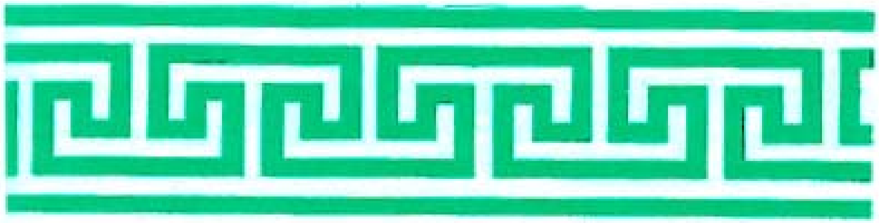
These methods can be used to create more complex patterns and borders, as well. For example, knowing how to create a perpendicular line to a segment, without extending the line past the segment, is necessary for this Islamic design (right):

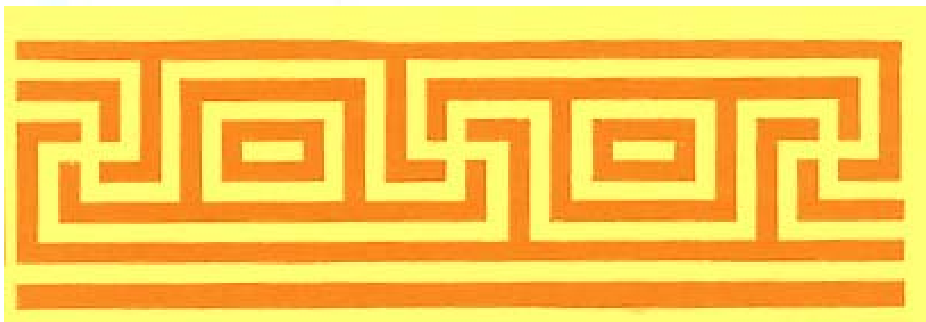


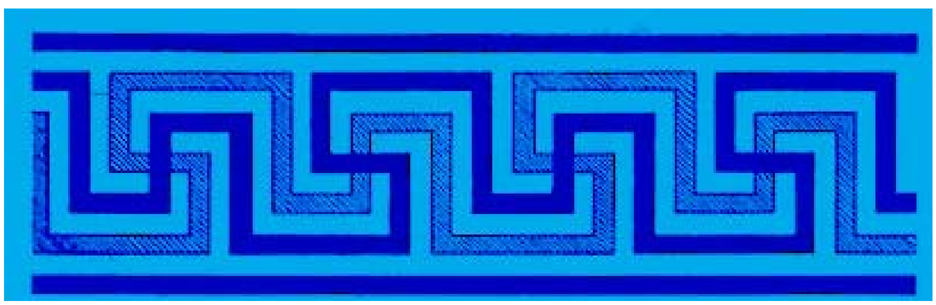


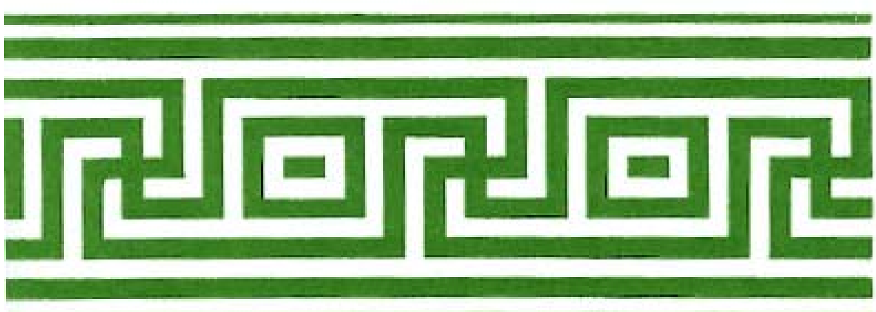
This design uses both the knowledge of how to create perpendicular lines and how to divide lines into equal segments. It also could have been constructed using circles and circumscribed squares, similar to the patterns created on page 16.

These borders are a further example of the intricate designs artists could create from geometry methods such as creating perpendicular lines and dividing segments into equal parts:









Islamic geometry was intimately tied to Islamic art, and Islamic geometry developed around the flat (2-dimensional) art of intricate and infinite designs and patterns. Muslim geometers like al-Farabi were very concerned with the practical applications of their work, as were algebraists like al-Khwārizmī. The development of Islamic mathematics both encouraged social stability and stimulated Islamic culture.